

Describing the asymptotic behaviour of multicolour Pólya urns via smoothing systems analysis

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Abstract

The present paper aims at describing in details the asymptotic composition of a class of d -colour Pólya urn: namely balanced, tenable and irreducible urns. We decompose the composition vector of such urns according to the Jordan decomposition of their replacement matrix. The projections of the composition vector onto the so-called *small* Jordan spaces are known to be asymptotically gaussian, but the asymptotic behaviour of the projections onto the *large* Jordan spaces are not known in full details up to now and are described by a limiting random variables called W , depending on the parameters of the urn.

We prove, via the study of smoothing systems, that the variable W has a density and that it is moment-determined.

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1 Introduction

A Pólya urn is a discrete time stochastic process which was originally introduced by Pólya and Eggenberger to model the spread of epidemics [EP23]. Since then, they have been useful in many different areas of mathematics and theoretical computer science and are therefore broadly studied. We can for example cite applications to the analysis of random trees (AVL [Mah98], 2–3 trees [FP05]), to the analysis of the Bandit algorithm [LPT04], or to the reinforced random walks (see for example Pemantle’s survey [Pem07]).

The range of methods used to study this random object is also very large. Historically studied by enumerative combinatorics (see for example [BP85]), Pólya urns have been efficiently studied by embedding in continuous time (or *poissonization*) since the works of Athreya and Karlin [AK68] (see for example [Jan04]). In parallel, since the seminal paper by Flajolet, Dumas & Puyhaubert [FDP06], the analytic combinatorics community successfully tackles the problem.

A Pólya urn process is defined as follows: an urn contains balls of different colours, let us denote by d the number of different colours available. Fix an initial composition $\alpha = {}^t(\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, meaning that there are α_i balls of colour i at time zero in the urn, for all $i \in \{1, \dots, d\}$. Fix a $d \times d$ matrix $R = (a_{i,j})_{1 \leq i,j \leq d}$ with integer coefficients. At each step of the process, pick up uniformly a ball at random in the urn, note its colour c ($c \in \{1, \dots, d\}$), put this ball back in the urn and add into the urn $a_{c,1}$ balls of colour 1, $a_{c,2}$ balls of colour 2, and so on, and so forth. The standard question asked is “how many balls of each colour are there in the urn” at time n ? when n tends to infinity?

Of course, the answer depends on the initial composition vector and on the replacement matrix chosen, and many different behaviours are exhibited in the literature. Different hypothesis are made in the literature, in order to control this behaviour:

- (T₋₁) The coefficients of R are all non-negative except the diagonal coefficients which can be equal to -1 .
- (T) The non-diagonal coefficients of R are non-negative, and, for all $i \in \{1, \dots, d\}$, either $a_{i,i} \geq -1$ or $-a_{i,i}$ is the gcd of $\{a_{1,i}, \dots, a_{d,i}, \alpha_i\}$.
- (B) The urn is balanced, meaning that there exists an integer S , called the balance, such that, for all $c \in \{1, \dots, d\}$, $\sum_{i=1}^d a_{c,i} = S$.

Assuming (T₋₁) or (T) permits to avoid non-tenable urns, i.e. urn schemes in which something impossible is asked: an example would be that you must subtract 3 balls of colour 1 from the urn while there is only 1 such ball in the urn. Allowing only non-negative coefficients in the replacement matrix and coefficients at least -1 on the diagonal permits to ensure that the urn is tenable. It is proven in [Pou08] that assumption (T) has the same effect. Some authors prefer working without such an assumption but then make all reasoning conditioned to tenability, i.e. conditioned on the event “no impossible configuration occur” (see for example [Jan04, Remark 4.2]).

The balance assumption (B) is quite standard in the literature, though it is not always necessary (this assumption is not done in [Jan04], for example). This assumption implies that the total number of balls in the urn is a deterministic function of time and this property is the foundation of combinatorics approaches while continuous time analysis of Pólya urns can be done for non-balanced urns.

Finally, we denote by (I) the assumption that the urn is irreducible, meaning that any colour can be produced from an initial composition with one unique ball: for a two-colour urn, being irreducible means having a non-triangular replacement matrix. The following definitions define the notion precisely.

Definition 1 (see [Jan04, page 4]). *Let $i, j \in \{1, \dots, d\}$, we say that i **dominates** j if an urn with initial composition \mathbf{e}_c and with replacement matrix R can possibly contain a ball of color i at some point, said differently, if there exists $n \geq 1$ such that $(R^n)_{i,j} > 0$. A color $c \in \{1, \dots, d\}$ is said to be **dominating** if it dominates every other color in $\{1, \dots, d\}$.*

Definition 2 (see [Jan04, page 4]). *We say that an urn of replacement matrix R is **irreducible** if and only if every colour is a dominating colour.*

Cases of non-irreducible urns are studied in the literature: the diagonal case $R = SI_d$ is the original Pólya-Eggenberger process and its behaviour is well described in [Ath69, BK64, JK77, CMP13], and triangular urn schemes are developped in [Jan06].

Remark that a large two-colour urn cannot have negative diagonal coefficients, whereas there exists d -colour Pólya urns with possibly negative coefficients having large eigenvalues. We thus have to include such urns in our study, that is why we only assume (T) and not the more restrictive assumption (T₋₁). In the present paper, we thus choose to assume (T), (B) and (I): we can cite many examples of urn processes that fall in this framework (see for example m -ary trees [CH01, CP04], paged binary trees [CH01], B -trees [CGPTT14]) and we will see all along the paper how each of these assumptions are used in the proofs. We are interested in the asymptotic behaviour of an urn under these three conditions.

The present setting is different from [Jan04]’s setting for d -colour Pólya urns, where (T₋₁) is assumed with further assumptions, but (B) is not: it is however mentioned in [Jan04, Remark 4.2] that Janson’s main results hold under our set of hypothesis. We will thus be able to apply Janson’s results before going further in the study of the asymptotic behaviour of the urn.

The following section is devoted to summarising the results of the literature and to introducing our framework.

2 Preliminaries

The behaviour of the urn process is already quite well known: We develop hereby the main results of the literature (mainly by Athreya [Ath69], Janson [Jan04] and Pouyanne [Pou08]) and thereafter introduce our main goal.

2.1 Jordan decomposition

Applying Perron-Frobenius theorem, in view of (B) and (I), gives that R admits S as a simple eigenvalue, and every other eigenvalue λ of R verifies $\text{Re}\lambda < S$. The balance hypothesis permits to apply Perron-Frobenius theorem and the irreducibility ensures that S is a simple eigenvalue of R . The matrix R can be written on its Jordan normal form, meaning that it is similar to a diagonal of blocks $\text{diag}(J_1, \dots, J_r)$ where each J_i is a matrix shaped as follows:

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix},$$

where λ is an eigenvalue of R . Note that several Jordan blocks can be associated to the same eigenvalue. In the following, we chose a Jordan block and study the behaviour of the projection of the composition vector onto the subspace associated to this Jordan block. Note that the fact that R is irreducible implies that S is a single eigenvalue of R .

Definition 3. *Let λ be an eigenvalue for R and $\sigma = \text{Re}\lambda/S$. We call λ a **large eigenvalue** of R if $1/2 < \sigma < 1$, or a **small eigenvalue** if $\sigma \leq 1/2$.*

A large Jordan block is a Jordan block of R associated with a large eigenvalue of R and a **small Jordan block** is a Jordan block associated with a small eigenvalue of R .

We denote by $U(n) \in \mathbb{N}^d$ the composition vector of the urn at time n : its i^{th} coordinate is by definition equal to the number of balls of colour i at time n in the urn. We are interested in the behaviour of $U(n)$ when n tends to infinity. It has been shown in the literature that $U(n)$ is easier to describe when decomposed according to the Jordan block decomposition of R . For every stable subspace E associated to a Jordan block of R , we will denote by π_E the projection on E and we will study separately each projection on a Jordan subspace E .

It is standard to embed urn processes in continuous time (see for example [AK68]): each ball is seen as a clock that rings after a random time with exponential law of parameter one, independently from other clocks in the urn. When a clock rings, it splits into $a_{i,j} + \delta_{i,j}$ balls of colour j ($\forall j \in \{1, \dots, d\}$) if the clock had colour i (recall that $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise). We denote by τ_n the time of the n^{th} ring in the urn and by $U^{CT}(t)$ the composition vector of the (continuous time) urn at time t . We have the following standard connexion: almost surely,

$$(U(n))_{n \geq 0} = (U^{CT}(\tau_n))_{n \geq 0}. \quad (1)$$

In addition, the process $(U(n))_{n \geq 0}$ is independent of the sequence of stopping times $(\tau_n)_{n \geq 0}$.

The asymptotic behaviour of the different projections of $U(n)$ and $U^{CT}(t)$ is partially described in the literature:

- In continuous time [Jan04],
 - small projections have a Gaussian behaviour, and
 - renormalised large projections converge to a random variable W^{CT} .
- In discrete time,
 - if R has only small eigenvalues apart from S , if E is the largest Jordan block associated to the eigenvalue λ realising the second highest real part (after S), and if $\sigma = \text{Re}\lambda/S = 1/2$ then projections onto E have a Gaussian behaviour [Jan04, Theorems 3.22 et 3.23]; and
 - renormalised large projections converge to a random variable W^{DT} [Pou08].

As one can see, the behaviour of small projections in discrete time is not known yet in full generality: Subsection 2.2 is devoted to describing the behaviour of $\pi_E(U(n))$ for all small Jordan space of R , and Subsection 2.3 will state the results concerning the projections of large Jordan spaces, as a preliminary to our main results.

2.2 Projections on small Jordan spaces

As explained above, the behaviour of the projections of the composition vector (in discrete time) onto the small Jordan blocks is not known yet in full generality. We sketch a proof of a general result in this subsection in order to complete the theory of Pólya urns under (B), (T) and (I).

Theorem 1. *If E is a block associated to a small eigenvalue λ of R , then there exists a covariance matrix Σ such that*

- If $\text{Re}\lambda = \frac{S}{2}$ then

$$\frac{\pi_E(U_{\alpha}(n))}{\sqrt{Sn \ln^{2\nu+1} n}} \rightarrow \mathcal{N}(0, \Sigma),$$

in distribution, asymptotically when n tends to infinity.

- If $\text{Re}\lambda < \frac{S}{2}$ then

$$\frac{\pi_E(U_{\alpha}(n))}{\sqrt{Sn}} \rightarrow \mathcal{N}(0, \Sigma),$$

in distribution, asymptotically when n tends to infinity.

Moreover, Σ does not depend on α .

Let E and λ as in Theorem 1. Let us first recall this result by Janson:

Theorem 2 ([Jan04, Theorem 3.15 (i) and (ii)]). *For all vector $\mathbf{b} \in \mathbb{R}^d$, define*

$$\tau_{\mathbf{b}}(n) = \inf\{t \geq 0 \mid \langle \mathbf{b}, U_{\alpha}^{CT}(t) \rangle \geq n\}.$$

Then,

(i) If $\operatorname{Re} \lambda = \frac{s}{2}$, then

$$\frac{1}{\sqrt{S n \ln^{2\nu+1} n}} \pi_E(U_{\alpha}^{CT}(\tau_{\mathbf{b}}(n))) \rightarrow \mathcal{N}(0, \sigma),$$

in distribution, where σ is a covariance matrix.

(ii) If $\operatorname{Re} \lambda < \frac{s}{2}$, then

$$\frac{1}{\sqrt{S n}} \pi_E(U_{\alpha}^{CT}(\tau_{\mathbf{b}}(n))) \rightarrow \mathcal{N}(0, \sigma),$$

n distribution, where σ is a covariance matrix.

Theorem 3.15 in [Jan04] is slightly different than the above version. The above version corresponds to the special case $z = n$ in Janson's Theorem 3.15. The (i) of Theorem 3.15 [Jan04] concerns the projection on the union of the small Jordan spaces, and it implies the (i) above by projection on a specified small Jordan space. The (ii) in Theorem 3.15 [Jan04] is more general than the above version, which is the special case $k = \nu$ of Janson's result. The matrix σ is given by Equations (3.11) and (2.15) in [Jan04] for case (i) above, and by Equations (3.12) and (2.16) in [Jan04]. It is important to note that σ does not depend on α .

Theorem 1 can be proved by using the *dummy* balls idea used in the proof of Theorems 3.21 and 3.22 in [Jan04]:

- Consider the continuous time urn with $d + 1$ colours, such that the $d + 1$ first colours evolve as the original d -colour process except that each time a ball splits, one ball of colour $d + 1$ is added to the process. When a ball of colour $d + 1$ splits, it splits into itself, adding no new balls in the process.
- Apply Theorem 2 to this $d + 1$ -colour process (this process is not irreducible, but Theorem 3.15 in [Jan04] applies also for non-irreducible processes).
- Go back to the original d -colour process by appropriate projection.

We do not develop the proof since no new idea is needed from there.

2.3 Projections on large Jordan spaces

Except for this digression on small eigenvalues, we are interested in the present paper in the behaviour of $U(n)$ along large Jordan blocks. **We will now on fix E a Jordan subspace of R associated to a large eigenvalue λ .** We denote by $\nu + 1$ the size of its associated Jordan block and we denote by v one eigenvector of E associated to the random variable λ . The asymptotic behaviour of $U(n)$ projected onto the subspace E is described by the following theorem:

Theorem 3 (cf. [Pou08]). *If $\frac{1}{2} < \sigma < 1$, then,*

$$\lim_{n \rightarrow \infty} \frac{\pi_E(U(n))}{n^{\lambda/s} \ln^{\nu} n} = \frac{1}{\nu!} W^{DT} v,$$

a.s. and in all L^p ($p \geq 1$), where $\pi_E(U(n))$ is the projection of the composition vector at time n onto E (according to the Jordan decomposition of R).

Remark: Note that different choices for v are possible, and that the random variable W^{DT} depends on this choice. The random variable W^{DT} should actually be denoted by $W_{E,v}^{DT}$ since it depends on the Jordan subspace E fixed and on the choice of v . For clarity's sake, we will stick to the ambiguous but simpler notation W^{DT} ; there is no ambiguity since E and v are fixed all along the present paper.

In continuous time, the behaviour of the composition vector projected on a large stable subspace verifies

Theorem 4 (see [Jan04]). *If $\frac{1}{2} < \sigma < 1$, then, almost surely,*

$$\lim_{t \rightarrow +\infty} \frac{\pi_E(U(t))}{t^{\nu} e^{\lambda t}} = \frac{1}{\nu!} W^{CT} v,$$

where π_E is the projection on the large stable subspace E . Moreover, the random variable W^{CT} admits moments of all orders.

Remark: Theorem 4 is proven in [Jan04] under slightly different hypothesis than ours. As explained in [Jan04, Remark 4.2], the result holds under (I), (T) and an additional assumption, namely

(PF) There exists an eigenvalue $\lambda_1 > 0$ of R such that, for all other eigenvalue λ of R , $\operatorname{Re} \lambda < \lambda_1$.

Assuming (B), (I) and (T) happen to imply (PF) through Perron-Frobenius theorem and Theorem 4 therefore hold in our setting.

We are interested in the two random variables W^{DT} and W^{CT} defined in Theorems 3 and 4. These random variables actually depend on the initial composition of the urn, denoted by $\alpha = {}^t(\alpha_1, \dots, \alpha_d)$ (meaning that there are, for all $i \in \{1, \dots, d\}$, α_i balls of type i in the urn at time 0). It is thus more rigorous to denote by W_α^{DT} (resp. W_α^{CT}) the random variable associated to the initial composition α , emphasizing that we have to study two infinite families of random variables.

Connexion (1) implies connexions between the random variables W induced by the discrete and continuous processes. We need the following result to deduce them:

$$\lim_{n \rightarrow \infty} n e^{-S\tau_n} = \xi, \quad (2)$$

almost surely, where ξ is a random variable with Gamma law of parameter $(\frac{\alpha_1 + \dots + \alpha_d}{S})$. This result is shown for two-colour urns in [CPS11], and can be straightforwardly adapted to the present case, using the balanced hypothesis (B). We do not develop this proof, which is very standard in the study of Yule processes (see for example [AN72, page 120]). We have

$$\frac{\pi_E(U^{CT}(\tau_n))}{\tau_n^\nu e^{\lambda\tau_n}} = \frac{\pi_E(U^{DT}(n))}{n^{\lambda/S} \ln^\nu n} \cdot \frac{n^{\lambda/S} \ln^\nu n}{\tau_n^\nu e^{\lambda\tau_n}}.$$

Moreover, Equation (2) implies $\frac{\ln n}{\tau_n} \rightarrow S$ when n tends to $+\infty$,

$$\frac{n^{\lambda/S} \ln^\nu n}{\tau_n^\nu e^{\lambda\tau_n}} = \left(\frac{\ln n}{\tau_n}\right)^\nu (n e^{-S\tau_n})^{\lambda/S} \rightarrow S^\nu \xi^{\lambda/S}.$$

Thus, for all initial composition α , we have (already mentioned in [Jan04]):

$$W_\alpha^{CT} \stackrel{(law)}{=} S^\nu \xi^{\lambda/S} W_\alpha^{DT}, \quad (3)$$

where ξ is a Gamma-distributed random variable with parameter $\frac{\alpha_1 + \dots + \alpha_d}{S}$, and where ξ and W^{DT} are independent.

We also have that $(U^{DT}(n(t)))_{t \geq 0} = (U^{CT}(t))_{t \geq 0}$ almost surely, where $n(t)$ is the total number of balls in the continuous time urn at time t . It implies that, for all initial composition α ,

$$W_\alpha^{DT} \stackrel{(law)}{=} S^{-\nu} \xi^{-\lambda/S} W_\alpha^{CT}, \quad (4)$$

where ξ is a Gamma-distributed random variable with parameter $\frac{\alpha_1 + \dots + \alpha_d}{S}$ but where ξ and W_α^{CT} are *not* independent, which can be verified via a covariance calculation.

The aim of the present paper is to gather information about W_α^{DT} and W_α^{CT} . Following the ideas developed in [CMP13] for the study of two Pólya urns, we will prove that, under (B), (I) and (T):

- Both W_α^{DT} and W_α^{CT} are moment-determined. Moreover, the Laplace transform of W_α^{DT} is convergent. See Theorem 7.
- Both W_α^{DT} and W_α^{CT} admit a density and their support is the whole complex plane if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the whole real line if $\lambda \in \mathbb{R}$. Moreover, the Fourier transform of W_α^{DT} is integrable. See Theorem 11.

These two results happen to be the first results on the variables W defined from a multi-colour urn (meaning with more than 2 colours). They are proven by using the same kind of arguments as in [CMP13] (induction reasoning for the moments and analysis of Fourier transforms for the densities): the main difficulties brought by the higher dimension come from the fact that the random variables are now complex and not necessarily real, and from the fact that the irreducibility condition is more

difficult to express and to handle in this context (recall that an irreducible two-colour urn is a non-triangular urn). Moreover, every calculation becomes more intricate.

The paper is organised as follows: Section 3 is devoted to model the continuous time urn process by a multitype branching process in order to characterise the random variables W^{CT} as solutions of systems of fixed point equations in law. In Section 4, we use these systems to study the moments of the W and prove that they are moment-determined, both in continuous and in discrete time. Section 5 deduce from the continuous time systems and from the moment study that W^{DT} are also determined as solution of a smoothing system. Finally, Section 6 contains the proof that the W admit a density, both in discrete and continuous time. It is very interesting to see how we will travel from discrete to continuous time all along the paper and how going from one world to the other is very fruitful.

3 Continuous time branching process – Smoothing system

In this section, we focus on the continuous-time process and show how to use the tree-like structure of the process. First, we reduce the study to only d initial compositions (instead of an infinite number), namely the initial compositions with a unique ball. Said differently, it is enough to study the random variables W_{e_1}, \dots, W_{e_d} where, for all $i \in \{1, \dots, d\}$, e_i is the vector whose coordinates are all 0 except the i^{th} which is 1 if $a_{i,i} \geq -1$ and $-a_{i,i}$ otherwise. We call e_1, \dots, e_d the **atomic initial compositions** of the urn. We then show, again using the tree-like structure of the process, that the random variables W_{e_1}, \dots, W_{e_d} verify a system of d smoothing equations.

First introduce further notations: For all $i \in \{1, \dots, d\}$, let us denote

$$\tilde{\alpha}_i = \begin{cases} \alpha_i & \text{if } a_{i,i} \geq -1, \\ \frac{\alpha_i}{-a_{i,i}} & \text{otherwise,} \end{cases} \quad (5)$$

and for all $i, c \in \{1, \dots, d\}$, let us denote

$$\tilde{a}_{c,i} = \begin{cases} a_{c,i} & \text{if } a_{i,i} \geq -1, \\ \frac{a_{c,i}}{-a_{i,i}} & \text{otherwise.} \end{cases} \quad (6)$$

In view of Assumption (T), for all $i, c \in \{1, \dots, d\}$, $\tilde{a}_{c,i}$ and $\tilde{\alpha}_i$ are integers.

Remark: If we suppose further that (T₋₁) holds, then $\tilde{\alpha}_i = \alpha_i$ for all $i \in \{1, \dots, d\}$ and $\tilde{a}_{i,j} = a_{i,j}$ for all $i, j \in \{1, \dots, d\}$.

3.1 Decomposition

To explain how to decompose the continuous time urn process, we will focus on an example of urn process, before generalizing to any urn process that verifies (B), (I) and (T). Assume for example that

$$R = \begin{pmatrix} 6 & 2 & 0 \\ 5 & -2 & 5 \\ 0 & 2 & 6 \end{pmatrix}.$$

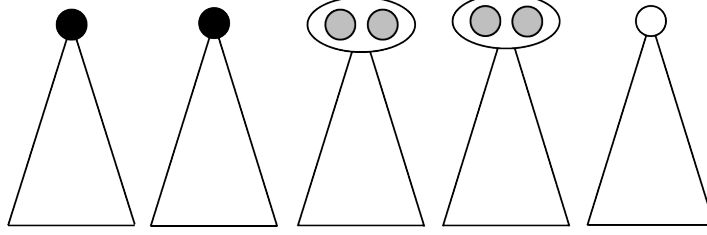
One can verify that the urn process defined by R verifies (B), (I) and (T). Moreover, its eigenvalues are 8, 6 and -4 . In particular, 6 is a large eigenvalue which permits us to apply Section and define W_{α}^{CT} through Theorem 4 for any initial composition α . In the following, we will denote by E_6 the Jordan stable subspace associated to this eigenvalue 6 and by π_6 the Jordan projection onto it.

Note that we can decompose the multitype branching process as shown in Figure 1, which gives the following

$$U_{(2,4,1)}^{CT} \stackrel{(law)}{=} \sum_{k=1}^2 U_{(1,0,0)}^{(k)} + \sum_{k=3}^4 U_{(0,2,0)}^{(k)} + \sum_{k=5}^5 U_{(0,0,1)}^{(k)},$$

where the $U^{(p)}$ are independent urn processes.

FIGURE 1 – *Decomposition of the urn process in continuous time – example.*



Let us quit the example and make the same reasoning as above in full generality under (B), (I) and (T). Recall that, for all $c \in \{1, \dots, d\}$, \mathbf{e}_c has all its coordinates equal to zero except for the c^{th} , which is equal to 1 if $a_{c,c} \geq 0$ and to $-a_{c,c}$ otherwise. We get

$$U_{\alpha}^{CT}(t) \stackrel{(law)}{=} \sum_{c=1}^d \sum_{p=\beta_{c-1}+1}^{\beta_c} U_{\mathbf{e}_c}^{(p)}(t),$$

where $\beta_0 = 0$ and $\beta_c = \sum_{j \leq c} \tilde{\alpha}_j$, and where the $U_{\mathbf{e}_c}^{(p)}(t)$ are independent copies of $U_{\mathbf{e}_c}^{CT}(t)$, independent of each other.

Dividing this equality in law by $t^\nu \mathbf{e}^{\lambda t}$, projecting onto E via π_E , and applying Theorem 4 gives

Theorem 5 (already mentioned in [Jan04, Remark 4.2]). *For all replacement matrix R and initial composition α verifying (B), (I) and (T),*

$$W_{\alpha}^{CT} \stackrel{(law)}{=} \sum_{c=1}^d \sum_{p=\beta_{c-1}+1}^{\beta_c} W_{\mathbf{e}_c}^{(p)},$$

where the $W_{\mathbf{e}_c}^{(p)}$ are independent copies of $W_{\mathbf{e}_c}^{CT}$, independent of each other and independent of U .

Theorem 5 allows to reduce the study to only d random variables, namely $(W_{\mathbf{e}_1}^{CT}, \dots, W_{\mathbf{e}_d}^{CT})$ instead of having to study an infinite family of such variables. Any information gathered about those d random variables will a priori give us some information about any W_{α}^{CT} .

3.2 Dislocation

In view of Theorem 5, it is enough to focus on the d atomic initial compositions $\mathbf{e}_1, \dots, \mathbf{e}_d$. Recall that for all $i \in \{1, \dots, d\}$, \mathbf{e}_i is the vector whose all components are zero, except the i^{th} which is equal to 1 if $a_{i,i} \geq 0$ and to $-a_{i,i}$ otherwise. We will now on denote by θ_i the non zero component of \mathbf{e}_i .

Let us again study first the particular example given by

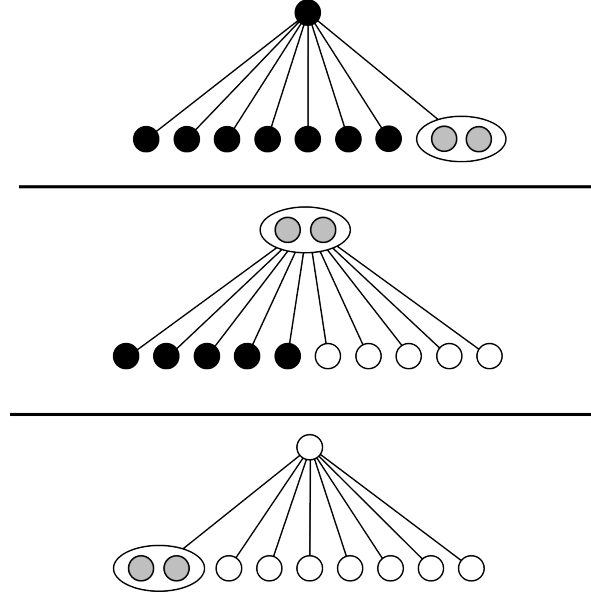
$$R = \begin{pmatrix} 6 & 2 & 0 \\ 5 & -2 & 5 \\ 0 & 2 & 6 \end{pmatrix}.$$

The three atomic initial compositions are given by $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 2, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, since $a_{1,1}, a_{3,3} \geq 0$ and since $a_{2,2} = -2$. In all three cases, the first step is deterministic: we know the colour of the first ball to be drawn and we therefore know what is the composition of the urn after the first split time (cf. Figure 2). We therefore have that,

$$U_{\mathbf{e}_1}^{CT} \stackrel{(law)}{=} \sum_{k=1}^7 U_{\mathbf{e}_1}^{(k)}(t - \tau^{(1)}) + \sum_{k=8}^8 U_{\mathbf{e}_2}^{(k)}(t - \tau^{(1)}),$$

$$U_{\mathbf{e}_2}^{CT} \stackrel{(law)}{=} \sum_{k=1}^5 U_{\mathbf{e}_1}^{(k)}(t - \tau^{(2)}) + \sum_{k=6}^{10} U_{\mathbf{e}_3}^{(k)}(t - \tau^{(2)}),$$

FIGURE 2 – Dislocation of a continuous time urn process – the different atomic initial compositions and their composition after the first drawing.



$$U_{\mathbf{e}_3}^{CT} \stackrel{(law)}{=} \sum_{k=1}^1 U_{\mathbf{e}_2}^{(k)}(t - \tau^{(3)}) + \sum_{k=2}^8 U_{\mathbf{e}_3}^{(k)}(t - \tau^{(3)}),$$

where the $U^{(k)}$ are independent continuous time urn processes with replacement matrix R , where $\tau^{(1)}$, $\tau^{(2)}$ and $\tau^{(3)}$ are independent random variables exponentially distributed of respective parameters 1, 2 and 1. The random variables $\tau^{(1)}$, $\tau^{(2)}$ and $\tau^{(3)}$ are the first split times that occur in urns of respective initial compositions \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 .

The same reasoning in full generality, for all replacement matrix R verifying (B), (I) and (T) gives that, for all $c \in \{1, \dots, d\}$,

$$U_{\mathbf{e}_c}^{CT} \stackrel{(law)}{=} \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} U_{\mathbf{e}_i}^{(k)}(t - \tau^{(c)}),$$

where $\gamma_0^{(c)} = 0$ and $\gamma_i^{(c)} = \sum_{j \leq i} \tilde{a}_{c,j} + \delta_{c,j}$, where $\tau^{(c)}$ is an exponentially distributed random variable of parameter θ_c , and where the $U^{(k)}$ are independent continuous time urn process with replacement matrix R . Dividing this equality in law by $t^\nu \mathbf{e}^{\lambda t}$, projecting onto E via π_E , and applying Theorem 4 gives

Theorem 6 (already mentioned in [Jan04, Theorem 3.9]). *Under assumptions (B), (I) and (T), for all $c \in \{1, \dots, d\}$,*

$$W_{\mathbf{e}_c}^{CT} \stackrel{(law)}{=} U^{\lambda/\theta_c} \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} W_{\mathbf{e}_i}^{(p)}, \quad (7)$$

where $\gamma_0^{(c)} = 0$, $\gamma_i^{(c)} = \sum_{j \leq i} (\tilde{a}_{c,j} + \delta_{c,j})$, where U is a uniform random variable on $[0, 1]$, and where the $W_{\mathbf{e}_i}^{(p)}$ are independent copies of $W_{\mathbf{e}_i}^{CT}$, independent of each other and of U .

Remark: If we assume (T₋₁) instead of (T) in the result above, we get

$$W_{\mathbf{e}_c}^{CT} \stackrel{(law)}{=} U^\lambda \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} W_{\mathbf{e}_i}^{(p)},$$

where $\gamma_0^{(c)} = 0$, $\gamma_i^{(c)} = \sum_{j \leq i} (a_{c,j} + \delta_{c,j})$, where U is a uniform random variable on $[0, 1]$, and where the $W_{\mathbf{e}_i}^{(p)}$ are independent copies of $W_{\mathbf{e}_i}^{CT}$, independent of each other and of U .

Remark: One can prove using Banach fixed point theorem in a well chosen space that the solution of System (7) is unique at fixed mean. A very similar proof is done in [Jan04, Proof of Theorem 3.9(iii), page 232–233].

4 Moments

This section is devoted to the study of the moments of the random variables $(W_{\mathbf{e}_i}^{DT})_{i \in \{1, \dots, d\}}$ and $(W_{\mathbf{e}_i}^{CT})_{i \in \{1, \dots, d\}}$. The convergence in all L^p ($p \geq 1$) stated in Theorems 3 and 4 ensures us that those random variables admit moments of all orders. Moreover,

Theorem 7. *Under assumptions (B), (I) and (T), for all initial composition α ,*

- (i) *the random variable W_{α}^{CT} is moment-determined.*
- (ii) *the Laplace series of W_{α}^{DT} has an infinite radius of convergence, which implies that W_{α}^{DT} is moment-determined.*

A similar result is already proved in [CMP13] in the two-colour case under assumptions (B), (I) and (T_{-1}) : remark that the above result extend [CMP13]’s result to a wider range of urn processes by assuming (T) instead of (T_{-1}) . Our proof is similar to the one developed there, slightly more involved due to the higher dimension of the system and to the slightly less restrictive tenability assumption.

The first step in the proof is the following lemma, which concerns the continuous time process: Equation 4 will then allow us to study the discrete time process.

Lemma 1. *If X_1, \dots, X_d are solution of System (7) and have moments of all orders, then the sequences $\left(\frac{\mathbb{E}|X_i|^p}{p! \ln^p p}\right)^{\frac{1}{p}}$, for all $i \in \{1, \dots, d\}$, are bounded.*

Proof. Let X_1, \dots, X_d solutions of System (7), let $\varphi(p) := \ln(p+2)$ and let, for all $i \in \{1, \dots, d\}$,

$$u_p^{(i)} := \frac{\mathbb{E}|X_i|^p}{p! \varphi(p)}.$$

Let us prove by induction on $p \geq 1$ that, for all $i \in \{1, \dots, d\}$, the sequence $\left(\frac{\mathbb{E}|X_i|^p}{p! \varphi(p)}\right)^{\frac{1}{p}}$ is bounded. Raise the equations of System (7) to power p . Since $\mathbb{E}|U^{\mu p}| = \frac{1}{p \operatorname{Re} \mu + 1}$ for all $\mu \in \mathbb{C}$ (with U uniformly distributed on $[0, 1]$), for all $c \in \{1, \dots, d\}$,

$$\begin{aligned} \mathbb{E}|X_c|^p &\leq \frac{1}{p \frac{\operatorname{Re} \lambda}{\theta_c} + 1} \left(\sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \mathbb{E}|X_i|^p \right. \\ &\quad \left. + \sum_{\substack{p_1 + \dots + p_{\gamma_d^{(c)}} = p \\ p_j \leq p-1}} \frac{p!}{p_1! \dots p_{\gamma_d^{(c)}}!} \prod_{i=1}^d \mathbb{E}|X_i|^{p_{\gamma_{i-1}^{(c)}+1}} \dots \mathbb{E}|X_i|^{p_{\gamma_i^{(c)}}} \right), \end{aligned}$$

which means,

$$p \frac{\operatorname{Re} \lambda}{\theta_c} \mathbb{E}|X_c|^p \leq \sum_{i=1}^d \tilde{a}_{c,i} \mathbb{E}|X_i|^p + \sum_{\substack{p_1 + \dots + p_{\gamma_d^{(c)}} = p \\ p_j \leq p-1}} \frac{p!}{p_1! \dots p_{\gamma_d^{(c)}}!} \prod_{i=1}^d \mathbb{E}|X_i|^{p_{\gamma_{i-1}^{(c)}+1}} \dots \mathbb{E}|X_i|^{p_{\gamma_i^{(c)}}}.$$

It implies that, for all $c \in \{1, \dots, d\}$,

$$p \frac{\operatorname{Re} \lambda}{\theta_c} u_p^{(c)} \leq \sum_{i=1}^d \tilde{a}_{c,i} u_p^{(i)} + \sum_{\substack{p_1 + \dots + p_{\gamma_d^{(c)}} = p \\ p_j \leq p-1}} \frac{\varphi(p_1) \dots \varphi(p_{\gamma_d^{(c)}})}{\varphi(p)} \prod_{i=1}^d u_{p_{\gamma_{i-1}^{(c)}+1}}^{(i)} \dots u_{p_{\gamma_i^{(c)}}}^{(i)} \quad (8)$$

Let

$$\Phi_c(p) := \sum_{\substack{p_1 + \dots + p_{\gamma_d^{(c)}} = p \\ p_j \leq p-1}} \frac{\varphi(p_1) \dots \varphi(p_{\gamma_d^{(c)}})}{\varphi(p)}. \quad (9)$$

A slight generalization of [CMP13, Lemma 1] ensures that

$$\Phi_c(p) \leq (1 + 8 \ln(p+2))^{\gamma_d^{(c)}},$$

for all $p \geq 2$, as soon as $\gamma_d^{(c)} \geq 1$. Remark that $\gamma_d^{(c)} = 0$ would mean that the urn with initial composition \mathbf{e}_c becomes empty after the first split time: this is impossible since we have assumed that the urn has a positive balance B (cf. Hypothesis (B)) – recall that if the balance is zero, then R have no large eigenvalue, and there is thus no W to study.

Denote by Δ_p the determinant of $p\text{Re}\lambda\Theta - \tilde{R}$ where $\Theta = \text{diag}\{\theta_1^{-1}, \dots, \theta_d^{-1}\}$ and $\tilde{R} = (\tilde{a}_{i,j})_{1 \leq i,j \leq d}$. This determinant is non-zero for all $p \geq 2$ since

$$\det(p\text{Re}\lambda\Theta - \tilde{R}) = \prod_{i=1}^d \theta_i^{-1} \det(p\text{Re}\lambda I_d - R)$$

and since $p\text{Re}\lambda > S$. Remark that, under (T_{-1}) , $\Theta = I_d$. In addition, let us denote by $\Delta_p(j, i)$ the determinant of $p\text{Re}\lambda\Theta - \tilde{R}$ in which the i^{th} column and the j^{th} line have been removed. For all $1 \leq i, j \leq d$, the polynomial $\Delta_p(j, i)$ has degree at most $d-1$ in p , which implies

$$\sup_{1 \leq i, j \leq d} \frac{|\Delta_p(j, i)|}{|\Delta_p|} = \mathcal{O}\left(\frac{1}{p}\right),$$

and there exists a constant $\eta > 0$ and an integer $p_0 \geq 1$ such that, for all $p \geq p_0$,

$$\sup_{1 \leq i, j \leq d} \frac{|\Delta_p(j, i)|}{|\Delta_p|} \leq \frac{\eta}{p}.$$

For all d -dimensional square real matrix M , we denote by $\|M\|_\infty = \sup_{1 \leq i, j \leq d} |M_{i,j}|$ and $\|M\| = \sup_{\|x\|_\infty=1} \|Mx\|_\infty$ (the norm $\|x\|_\infty$ of a vector x being the maximum of the modulus of its coordinates) respectively the infinite norm and the operator norm of M . Of course, these two norms are equivalent and there exists a constant $\kappa > 0$ such that, for all d -dimensional square real matrix M ,

$$\|M\| \leq \kappa \|M\|_\infty.$$

Let us denote by $\Delta_p(c)$ the determinant of the matrix $p\text{Re}\lambda\Theta - \tilde{R}$ in which the c^{th} column has been replaced by a column of 1. We know that Δ_p has degree d in p whereas $\Delta_p(c)$ is a polynomial with degree at most $d-1$ in p . It implies that there exists an integer $p_1 \geq p_0$ such that, for all $p \geq p_1$, for all $c \in \{1, \dots, d\}$,

$$\frac{\Delta_p(c)}{\Delta_p} (1 + 8 \ln(p+2))^{\gamma_d^{(c)}} \leq \frac{1}{2\kappa^2 \eta \text{Re}\lambda}. \quad (10)$$

Finally, since $\frac{\|p\text{Re}\lambda\Theta - \tilde{R}\|_\infty}{p^{\frac{\text{Re}\lambda}{\theta_c}}} \rightarrow 1$ when p tends to $+\infty$, there exists $p_2 \geq p_1$ such that, for all $p \geq p_2$,

$$\frac{\|p\text{Re}\lambda\Theta - \tilde{R}\|_\infty}{p^{\frac{\text{Re}\lambda}{\theta_c}}} \leq 2.$$

Let us define

$$A := \max\{(u_q^{(i)})^{\frac{1}{q}}, 1 \leq q \leq p_2, 1 \leq i \leq d\},$$

and prove by induction on $p \geq p_2$ that, for all $q \leq p$, for all $c \in \{1, \dots, d\}$, $(u_q^{(c)})^{\frac{1}{q}} \leq A$. Assume that this is true for all $p \geq p_2$. Equation (8) implies

$$p\text{Re}\lambda\Theta u_p^{(c)} \leq \sum_{i=1}^d \tilde{a}_{c,i} u_p^{(i)} + A^p \Phi(p).$$

Let v_1, \dots, v_d the solution of system

$$p\text{Re}\lambda\Theta v_c = \sum_{i=1}^d \tilde{a}_{c,i} v_i + A^p \Phi(p).$$

By resolving this Cramér system, we get, in view of Equation (10),

$$v_c = A^p \Phi(p) \frac{\Delta_p(c)}{\Delta_p} \leq \frac{A^p}{2\kappa^2 \eta \text{Re}\lambda}.$$

For all $p \geq p_2$,

$$(p\text{Re}\lambda\Theta - \tilde{R})\mathbf{u}^{(p)} \leq (p\text{Re}\lambda\Theta - \tilde{R})\mathbf{v} \leq \|p\text{Re}\lambda\Theta - \tilde{R}\| A^p \frac{1}{2\kappa^2 \eta \text{Re}\lambda} \boldsymbol{\omega},$$

where $\mathbf{u}^{(p)}$ and \mathbf{v} denote the vectors of respective coordinates $(u_p^{(i)})_{1 \leq i \leq d}$ and $(v_i)_{1 \leq i \leq d}$, where $\boldsymbol{\omega}$ is the vector whose all coordinates are equal to 1, and where the sign \leq between two vectors is to be read coordinate by coordinate. Consider the infinite norm $\|\cdot\|_\infty$ on \mathbb{R}^d :

$$\|(p\text{Re}\lambda\Theta - \tilde{R})\mathbf{u}^{(p)}\|_\infty \leq A^p \frac{\|p\text{Re}\lambda\Theta - \tilde{R}\|}{2\kappa^2 \eta \text{Re}\lambda} \leq A^p \frac{\|p\text{Re}\lambda\Theta - \tilde{R}\|_\infty}{2\kappa \eta \text{Re}\lambda} \leq A^p \frac{p}{\kappa \eta}.$$

Let us denote $M = p\text{Re}\lambda\Theta - \tilde{R}$. The coefficients of M^{-1} are given by

$$(M^{-1})_{i,j} = (-1)^{i+j} \frac{\Delta_p(j, i)}{\Delta_p},$$

where Δ_p is the detreminant of M , and $\Delta_p(j, i)$ is the determinant of the matrix M in which the j^{th} line and the i^{th} column have been removed. By definition of p_2 , for all $p \geq p_2$,

$$\|M^{-1}\|_\infty = \sup_{1 \leq i, j \leq d} |(M^{-1})_{i,j}| \leq \frac{\eta}{p},$$

which implies, for all $p \geq p_2$,

$$\|\mathbf{u}^{(p)}\|_\infty \leq \|M^{-1}\|_\infty A^p \frac{p}{\kappa \eta} \leq \kappa \|M^{-1}\|_\infty A^p \frac{p}{\kappa \eta} \leq A^p.$$

Finally, for all $c \in \{1, \dots, d\}$,

$$u_c^{(p)} \leq A^p,$$

which concludes the proof. \square

Proof of Theorem 7. (i) Lemma 1 implies that, for all $i \in \{1, \dots, d\}$, the random variable $W_{\mathbf{e}_i}^{CT}$, which admits moments of all orders in view of Theorem 4 verifies Carleman's criterion. Theorem 5 allows eventually to generalize Lemma 1 to any initial composition: For all initial composition $\boldsymbol{\alpha}$, $W_{\boldsymbol{\alpha}}^{CT}$ also verifies the Carleman's criterion.

(ii) We have the following inequality: there exists a constant C such that, for all integer p , for any initial composition,

$$\frac{\mathbb{E}|W^{CT}|^p}{p!} \leq C^p \ln^p p.$$

It implies, via Equation (3), there exists a constant D such that, for all integer p ,

$$\frac{\mathbb{E}|W^{DT}|^p}{p!} \leq D^p \frac{\ln^p p}{\Gamma\left(\frac{p\text{Re}\lambda+1}{S}\right)},$$

which implies that the Laplace series of W^{DT} has an infinite radius of convergence. \square

Theorem 7 gives an upper bound for the moments of W^{CT} and W^{DT} ; note that no lower bound is known up to now.

5 Discrete time urn process – Smoothing system

5.1 Smoothing system in discrete time

This subsection is devoted to deduce from Parts 3 and 4 that the random variables $W_{\mathbf{e}_1}^{DT}, \dots, W_{\mathbf{e}_d}^{DT}$ are solution of a smoothing system:

Theorem 8. *Under assumptions (B), (T) and (I), for every colour $c \in \{1, \dots, d\}$,*

$$W_{\mathbf{e}_c}^{DT} \stackrel{(law)}{=} \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_k^{(c)}\right)^{\lambda/S} W_{\mathbf{e}_i}^{(k)}, \quad (11)$$

where $\gamma_0 = 0$ and $\gamma_i^{(c)} = \sum_{j=1}^i (\tilde{a}_{c,j} + \delta_{c,j})$ for all $i \in \{1, \dots, d\}$; where the $W_{\mathbf{e}_i}^{(k)}$ are independent copies of $W_{\mathbf{e}_i}^{DT}$, independent of each other; and where $V^{(c)} = (V_1, \dots, V_{\gamma_d^{(c)}})$ is a Dirichlet-distributed random vector independent of the W and of parameter $\boldsymbol{\pi}$, given by

$$\pi_k = \theta_i/S \quad \text{if} \quad \gamma_{i-1}^{(c)} < k \leq \gamma_i^{(c)}.$$

Remark: If we assume (T_{-1}) instead of (T) in the theorem above, we obtain the following system:

$$W_{\mathbf{e}_c}^{DT} \stackrel{(law)}{=} \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} V_k^{\lambda/S} W_{\mathbf{e}_i}^{(k)},$$

where $\gamma_0 = 0$ and $\gamma_i^{(c)} = \sum_{j=1}^i (a_{c,j} + \delta_{c,j})$ for all $i \in \{1, \dots, d\}$; where the $W_{\mathbf{e}_i}^{(k)}$ are independent copies of $W_{\mathbf{e}_i}^{DT}$, independent of each other; and where $V = (V_1, \dots, V_{S+1})$ is a Dirichlet-distributed random vector of parameter $(\frac{1}{S}, \dots, \frac{1}{S})$, independent of the W .

Proof. Let us prove that, for all $p \geq 1$,

$$\mathbb{E}|W_{\mathbf{e}_c}^{DT}|^p = \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} V_k^{\lambda/S} W_{\mathbf{e}_i}^{(k)} \right|^p.$$

Since, $W_{\mathbf{e}_c}^{DT}$ is moment-determined in view of Theorem 7, this will conclude the proof. Let us use Connexion (1), which gives, for all $c \in \{1, \dots, d\}$,

$$\mathbb{E}|W_{\mathbf{e}_c}^{CT}|^p = S^{p\nu} \mathbb{E}|\xi_c^{p\lambda/S}| \mathbb{E}|W_{\mathbf{e}_c}^{DT}|^p, \quad (12)$$

where ξ_c is a Gamma-distributed random variable, of parameter θ_c/S .

Let V_c a Beta-distributed random variable of parameter $(\theta_c/S, 1)$. Remark that if U is uniformly distributed on $[0, 1]$, we have

$$V_c \stackrel{(law)}{=} U^{S/\theta_c}$$

For all $i \in \{1, \dots, d\}$, for all $k \geq 1$, $\zeta_{i,k}$ is a Gamma-distributed random variable of parameter θ_c/S . Thus, the random variable

$$\zeta_c = V_c \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \zeta_{i,k}$$

is Gamma distributed with parameter θ_i/S (it can be verified by calculating its moments). Finally, let

$$V_{i,k}^{(c)} = \frac{\zeta_{i,k}}{\sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \zeta_{i,k}}.$$

Then (see for example [Ber06, Lemma 2.2]), the $\gamma_d^{(c)}$ -dimensional random vector $V^{(c)}$ whose coordinates are given by

$$V_k^{(c)} = V_{i,k}^{(c)} \quad \text{if} \quad \gamma_{i-1}^{(c)} < k \leq \gamma_i^{(c)}$$

is Dirichlet-distributed of parameter $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{\gamma_d^{(c)}})$, where

$$\pi_k = \theta_i/S \quad \text{if} \quad \gamma_{i-1}^{(c)} < k \leq \gamma_i^{(c)},$$

and independent from ζ_c . Thus, System (7) together with Equation 12, gives that, for all $c \in \{1, \dots, d\}$,

$$\begin{aligned} S^{p\nu} \mathbb{E}[\zeta_c^{p\lambda/S}] \mathbb{E}[W_{\mathbf{e}_c}^{DT}]^p &= \mathbb{E} \left| V_c^{\lambda/S} \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} W_{\mathbf{e}_i}^{CT,(k)} \right|^p \\ &= \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} V_c^{\lambda/S} S^\nu \zeta_{i,k}^{\lambda/S} W_{\mathbf{e}_i}^{DT,(k)} \right|^p. \end{aligned}$$

On the other hand, by independence of $V^{(c)}$ and ζ_c ,

$$\begin{aligned} \mathbb{E}[\zeta_c^{p\lambda/S}] \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_{i,k}^{(c)} \right)^{\lambda/S} W_{\mathbf{e}_i}^{DT,(k)} \right|^p &= \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(\zeta_c V_{i,k}^{(c)} \right)^{\lambda/S} W_{\mathbf{e}_i}^{DT,(k)} \right|^p \\ &= \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} (\zeta_{i,k} V_c)^{\lambda/S} W_{\mathbf{e}_i}^{DT,(k)} \right|^p, \end{aligned}$$

which implies

$$\mathbb{E}[W_{\mathbf{e}_c}^{DT}]^p = \mathbb{E} \left| \sum_{i=1}^d \sum_{k=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_{i,k}^{(c)} \right)^{\lambda/S} W_{\mathbf{e}_i}^{DT,(k)} \right|^p,$$

for all $p \geq 0$, which concludes the proof since W^{DT} is moment-determined. \square

5.2 Contraction

The main goal of this section is to prove that the solution of System (11) is unique. We therefore use the so-called contraction method. This method, presented for example in Neininger-Rüschendorf's survey [NR06] consists in applying the Banach fixed point theorem in an appropriate complete Banach space. It has already been used in a Pólya urn context in the literature. In [KN13] the contraction method is used as a new approach to prove an equivalent of Theorem 3 for large and small eigenvalues (in discrete time). In [CMP13], it is used as in the present paper, to prove the unicity of the solution of a two-equation system, in the study of large two-colour Pólya urns. In [Jan04], it is also used to prove the unicity of the solution of system (7): therefore, we will only develop the proof for the discrete case. Similar proofs can be found in [KN13] or [CMP13].

Let \mathcal{M}_2 be the space of complex square integrable probability measures. For all $A \in \mathbb{C}$, let $\mathcal{M}_2^{\mathbb{C}}(A)$ be the subspace of measures in \mathcal{M}_2 with mean A . We consider the Wasserstein distance as follow: for all μ, ν two measures in $\mathcal{M}_2^{\mathbb{C}}(A)$,

$$d_W(\mu, \nu) = \min_{X \sim \mu, Y \sim \nu} \|X - Y\|_2,$$

where $\|\cdot\|_2$ is the L^2 -norm on \mathbb{C} .

For all $A_1, \dots, A_d \in \mathbb{C}$, let us define the Wasserstein distance on $\times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ as follow: for all $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d)$ two elements of $\times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$,

$$d(\boldsymbol{\mu}, \boldsymbol{\nu}) = \max_{1 \leq i \leq d} \{d_W(\mu_i, \nu_i)\}.$$

We know that $(\mathcal{M}_2^{\mathbb{C}}(A), d_W)$ and thus $\times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ are complete metric spaces (see for example [Dud02]).

The random vector $(W_{\mathbf{e}_1}^{DT}, \dots, W_{\mathbf{e}_d}^{DT})$ is solution of System (11):

$$W_{\mathbf{e}_c} \stackrel{(law)}{=} \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)}\right)^{\lambda/S} W_{\mathbf{e}_i}^{(p)}.$$

For all $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(m_i)$, for all $c \in \{1, \dots, d\}$, let

$$K_c(\boldsymbol{\mu}) = \mathcal{L} \left(\sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)}\right)^{\lambda/S} x_i^{(p)} \right),$$

where $\gamma_0 = 0$, $\gamma_i^{(c)} = \sum_{j \leq i} (\tilde{a}_{c,j} + \delta_{c,j})$ and for all $i \in \{1, \dots, d\}$, the $(x_i^{(p)})_{1 \leq p \leq S+1}$ are independent random variables with law μ_i , independent of each other and of vector V , which is Dirichlet-distributed with parameter $(1/S, \dots, 1/S)$. We define the function K as

$$K(\boldsymbol{\mu}) = (K_1(\boldsymbol{\mu}), \dots, K_d(\boldsymbol{\mu})),$$

and prove the following result via Banach fixed point theorem:

Proposition 1. *For all large eigenvalue λ of the replacement matrix R , for all $\mathbf{A} = (A_1, \dots, A_d) \in \mathbb{C}^d$, denote by $\Psi(\mathbf{A})$ the vector whose coordinates are given by*

$$\Psi(\mathbf{A})_i = \frac{A_i}{\lambda + \theta_i} \quad \text{for all } i \in \{1, \dots, d\}.$$

(i) *For all $\mathbf{A} = (A_1, \dots, A_d) \in \mathbb{C}^d$ such that $\Psi(\mathbf{A}) \in \text{Ker}(R - \lambda I_d)$, the function K is an application from $\times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ into itself.*

(ii) *Moreover, the law of $(W_{\mathbf{e}_1}^{DT}, \dots, W_{\mathbf{e}_d}^{DT})$ is the unique solution of (11) at fixed mean.*

Remark: If we assume (T_{-1}) in addition, then, remark that $\Psi(\mathbf{A}) \in \text{Ker}(R - \lambda I_d)$ if and only if $\mathbf{A} \in \text{Ker}(R - \lambda I_d)$.

Proof. (i) First remark that

$$\mathbb{E} K_c(\boldsymbol{\mu}) = \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \mathbb{E} \left(V_p^{(c)} \right)^{\lambda/S} \mathbb{E} x_i^{(p)}$$

because (V_1, \dots, V_{S+1}) is independent of $(x_i^{(1)}, \dots, x_i^{(S+1)})_{1 \leq i \leq d}$. Since, for all $p \in \{1, \dots, S+1\}$, for all $i \in \{1, \dots, d\}$, $\mathbb{E} x_i^{(p)} = A_i$, we have

$$\begin{aligned} \mathbb{E} K_c(\boldsymbol{\mu}) &= \sum_{i=1}^d A_i \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \mathbb{E} \left(V_p^{(c)} \right)^{\lambda/S} \\ &= \sum_{i=1}^d A_i \frac{\gamma_i^{(c)} - \gamma_{i-1}^{(c)}}{1 + \lambda \theta_i^{-1}} = \sum_{i=1}^d A_i \frac{\tilde{a}_{c,i} + \delta_{c,i}}{1 + \lambda \theta_i^{-1}} \\ &= \sum_{i=1}^d A_i \frac{a_{c,i} + \delta_{c,i} \theta_i}{\theta_i + \lambda} = \sum_{i=1}^d (a_{c,i} + \delta_{c,i} \theta_i) B_i, \end{aligned}$$

where $\mathbf{B} = \Psi(\mathbf{A})$, and where U is a uniformly distributed random variable on $[0, 1]$. The above calculations are true because $\left(V_p^{(c)}\right)^{\lambda/S} \stackrel{(law)}{=} U^{\lambda/\theta_c}$ for all $p \in \{1, \dots, S+1\}$, and because $\text{Re} \lambda > S/2$, which implies $\lambda \neq -\theta_c$. Since λ is an eigenvalue of R , and $\mathbf{B} = (B_1, \dots, B_d) \in \text{Ker}(R - \lambda I_d)$, we have

$$\sum_{i=1}^d a_{c,i} B_i = \lambda B_c$$

for all $1 \leq c \leq d$. It implies

$$\mathbb{E}K_c(\boldsymbol{\mu}) = (\lambda + \theta_c)B_c = A_c$$

for all $\boldsymbol{\mu} \in \times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$. Moreover $K(\boldsymbol{\mu})$ is square-integrable, which implies that K is indeed a function from $\times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ into itself, for all \mathbf{A} such that $\Psi(\mathbf{A}) \in \text{Ker}(R - \lambda I_d)$.

(ii) Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d) \in \times_{i=1}^d \mathcal{M}_2^{\mathbb{C}}(A_i)$ be two solutions of System 11, meaning that

$$K\boldsymbol{\mu} = \boldsymbol{\mu} \quad \text{and} \quad K\boldsymbol{\nu} = \boldsymbol{\nu}.$$

Let us prove that $d(\boldsymbol{\mu}, \boldsymbol{\nu}) = 0$, using the total variance law: it is enough to prove that, for all $i \in \{1, \dots, d\}$, $d_W(\mu_i, \nu_i) = 0$. Let $(x_1^{(p)}, \dots, x_d^{(p)})_{1 \leq p \leq S+1}$ be a sequence of random variables having the same law, namely $\boldsymbol{\mu}$, and $(y_1^{(p)}, \dots, y_d^{(p)})_{1 \leq p \leq S+1}$ a sequence of random variables with the same law $\boldsymbol{\nu}$, and let $(V_1^{(c)}, \dots, V_{S+1}^{(c)})$ be a Dirichlet-distributed random vector of parameter $\boldsymbol{\pi}$, where

$$\pi_k = \theta_i/S \quad \text{if } \gamma_{i-1}^{(c)} < k \leq \gamma_i^{(c)}.$$

We thus have, for all $c \in \{1, \dots, d\}$,

$$\begin{aligned} d_W(\mu_c, \nu_c)^2 &= d_W(K_c(\boldsymbol{\mu}), K_c(\boldsymbol{\nu}))^2 \\ &\leq \left\| \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)}\right)^{\lambda/S} (x_i^{(p)} - y_i^{(p)}) \right\|_2^2 \\ &= \mathbb{E} \left| \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)}\right)^{\lambda/S} (x_i^{(p)} - y_i^{(p)}) \right|^2 \\ &= \mathbb{E} \left[\mathbb{E} \left(\left| \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)}\right)^{\lambda/S} (x_i^{(p)} - y_i^{(p)}) \right|^2 \mid (V_1, \dots, V_{S+1}) \right) \right]. \end{aligned}$$

Since (V_1, \dots, V_{S+1}) is independent of $(x_1^{(p)}, \dots, x_d^{(p)})_{1 \leq p \leq d}$, we get:

$$\begin{aligned} d_W(K_c(\boldsymbol{\mu}), K_c(\boldsymbol{\nu}))^2 &\leq \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \mathbb{E} \left| \left(V_p^{(c)}\right)^{2\lambda/S} \right| \mathbb{E} |x_i^{(p)} - y_i^{(p)}|^2 \\ &\leq \sum_{i=1}^d \mathbb{E} |x_i - y_i|^2 \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \mathbb{E} \left| \left(V_p^{(c)}\right)^{2\lambda/S} \right| \\ &= \sum_{i=1}^d \frac{\gamma_i^{(c)} - \gamma_{i-1}^{(c)}}{2\text{Re}\lambda\theta_i^{-1} + 1} \|x_i - y_i\|_2^2, \end{aligned}$$

for all $(x_1, \dots, x_d) \sim \boldsymbol{\mu}$ and $(y_1, \dots, y_d) \sim \boldsymbol{\nu}$. Thus,

$$d_W(K_c(\boldsymbol{\mu}), K_c(\boldsymbol{\nu}))^2 \leq \sum_{i=1}^d d_W(\mu_i, \nu_i)^2 \frac{\tilde{a}_{c,i} + \delta_{c,i}}{2\text{Re}\lambda\theta_i^{-1} + 1},$$

which implies that

$$2\text{Re}\lambda\Delta_c \leq \sum_{i=1}^d a_{i,c}\Delta_i,$$

where $\Delta_i = \frac{d_W(\mu_i, \nu_i)^2}{\theta_i + 2\text{Re}\lambda}$, for all $i \in \{1, \dots, d\}$. For all $c \in \{1, \dots, d\}$,

$$2\text{Re}\lambda\Delta_c \leq (R\Delta)_c.$$

Let $\mathbf{v} = (v_1, \dots, v_d)$ be a horizontal vector with positive entries such that $\mathbf{v}R = S\mathbf{v}$, then

$$\mathbf{v}(R\Delta) = S\mathbf{v}\Delta.$$

The existence of such a \mathbf{v} is a consequence of the irreducibility of the urn, namely hypothesis (I), as explained in [Jan04]. It implies that,

$$S \sum_{c=1}^d v_c \Delta_c = \sum_{c=1}^d v_c (R\Delta)_c \geq 2\operatorname{Re}\lambda \sum_{c=1}^d v_c \Delta_c.$$

Since $\operatorname{Re}\lambda/S > 1/2$, this last inequality implies that

$$\sum_{c=1}^d v_c \Delta_c = 0,$$

and thus, by positivity of the v_c and non negativity of the Δ_c , we get, for all $c \in \{1, \dots, d\}$, $\Delta_c = 0$. It thus implies that, for all $c \in \{1, \dots, d\}$,

$$\mu_c = \nu_c, \quad \text{and thus} \quad \boldsymbol{\mu} = \boldsymbol{\nu},$$

which concludes the proof. □

5.3 Decomposition in discrete time

The same argument as used to prove Theorem 8 can be use to prove the following result from Theorem 5 and Theorem 7:

Theorem 9. *Under assumptions (B), (T) and (I), for all initial composition $\boldsymbol{\alpha}$,*

$$W_{\boldsymbol{\alpha}}^{DT} \stackrel{(law)}{=} \sum_{c=1}^d \sum_{p=\beta_{c-1}+1}^{\beta_c} Z_p^{N/S} W_{\mathbf{e}_c}^{(p)},$$

where $\beta_0 = 0$, $\beta_i = \sum_{j \leq i} \tilde{\alpha}_j$; where the $W_{\mathbf{e}_c}^{(p)}$ are independent copies of $W_{\mathbf{e}_c}^{DT}$, independent of each other; and where $Z = (Z_1, \dots, Z_{\beta_d})$ is a Dirichlet-distributed random vector independent of the W and of parameter $\boldsymbol{\eta}$ given by

$$\boldsymbol{\eta}_k = \theta_i/S \text{ if } \beta_{i-1} < k \leq \beta_i.$$

Remark: If we assume (T_{-1}) instead of (T) in the theorem above, we obtain the following system: for all initial composition $\boldsymbol{\alpha}$,

$$W_{\boldsymbol{\alpha}}^{DT} \stackrel{(law)}{=} \sum_{c=1}^d \sum_{p=\beta_{c-1}+1}^{\beta_c} Z_p^{N/S} W_{\mathbf{e}_c}^{(p)},$$

where $\beta_0 = 0$, $\beta_i = \sum_{j \leq i} \alpha_j$; where $Z = (Z_1, \dots, Z_{\beta_d})$ is a Dirichlet-distributed random vector of parameter $(\frac{1}{S}, \dots, \frac{1}{S})$; and where the $W_{\mathbf{e}_c}^{(p)}$ are independent copies of $W_{\mathbf{e}_c}^{DT}$, independent of each other and of Z .

5.4 Tree structure in discrete time

Theorems 9 and 8 can be proven from scratch by an analogue of the proof of Theorems 5 and 6: i.e. using the underlying tree structure of the urn. The analysis of the tree structure in discrete time is however more intricate since the subtrees of the considered forest are not independent. Such a tree decomposition in discrete time is already proposed in [CMP13] (for two-colour urns) or [KN13] but both paper assume (B), (I) and (T_{-1}) . We will develop this alternative proof of Theorem 9 because it happens to be more complicated due to the possibly negative diagonal coefficient of the replacement matrix.

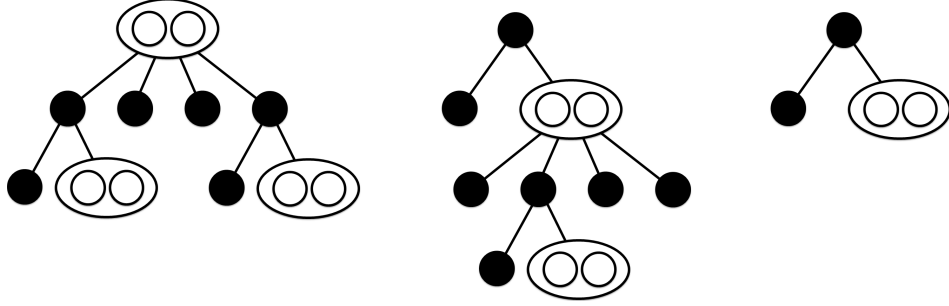
Alternative proof of Theorem 9. The discrete time urn process can be seen as a forest whose leaves can be of d different colours: we refer to Figure 3¹. At time zero, the forest is composed of $\tilde{\alpha}_i$ roots of

¹Figure 3 is an example of a two-colour urn. The example taken is chosen for its simplicity even if the chosen urn is a small urn. It does not affect the arguments developed in the proof.

FIGURE 3 – A realisation at time $n = 7$ of the forest associated to the urn process

with initial composition ${}^t(2, 2)$ and with replacement matrix $R = \begin{pmatrix} -2 & 4 \\ 2 & 0 \end{pmatrix}$.

Remark that $S_1(7) = 3$, $S_2(7) = 3$ and $S_3(7) = 1$.
Moreover, $D_1(7) = 8$, $D_2(7) = 3$ and $D_3(7) = 1$.



colour i (for all $i \in \{1, \dots, d\}$), each of these roots contains θ_i balls of colour i . At each step, we pick up uniformly at random a ball: this ball belongs to a leaf. The picked leaf then becomes an internal node, which has $\gamma_d^{(c)}$ children, amongst them $\tilde{a}_{i,j} + \delta_{i,j}$ contain θ_j balls of colour j (for all $j \in \{1, \dots, d\}$) if the picked leaf was of colour i .

The composition of the urn is thus described by the set of leaves of the forest. Let us number the subtrees of the forest from trees rooted by colour 1 up to trees rooted by colour d . If we denote by $D_p(n)$ the number of balls in leaves of the p^{th} subtree of the forest, then this p^{th} subtree of the forest at time n represents the composition vector of an urn process with initial composition of cardinal θ_c if $\beta_{c-1} < p \leq \beta_c$, taken at *internal time* $S_p(n) = \frac{D_p(n) - \theta_p}{S}$. Indeed, the *internal time* in the p^{th} tree is actually the number of its internal nodes; the fact that the urn is balanced means that all internal node of the tree has given birth to $S + 1$ balls, which give the above relationship between leaves and internal nodes in the p^{th} subtree.

We thus have

$$U_{\alpha}(n) \stackrel{(law)}{=} \sum_{c=1}^d \sum_{p=\beta_{c-1}+1}^{\beta_c} U_{e_c}^{(p)} \left(\frac{D_p(n) - \omega_p}{S} \right), \quad (13)$$

where $\beta_0 = 0$, where for all $c \geq 1$, $\beta_c = \sum_{i=1}^c \tilde{a}_i$; where $\omega_p = \theta_i$ if and only if $\beta_{i-1} < p \leq \beta_i$; and where the urn processes $U_{e_c}^{(p)}$ are independent copies of the process U_{e_c} , independent of each other.

We are thus interested in the asymptotic behaviour of the vector $(D_1(n), \dots, D_{\beta_d}(n))$ when n grows to infinity. This vector actually happens to be the composition vector of a $(\beta_d = \sum_{i=1}^d \tilde{a}_i)$ -colour Pólya urn with initial composition $\omega = {}^t(\omega_1, \dots, \omega_{\beta_d})$, where

$$\omega_k = \theta_i \quad \text{if } \beta_{i-1} < k \leq \beta_i.$$

Indeed, forget the initial colouring of the leaves and colour the leaves of the i th subtree with colour i ; at each step, a leaf of the forest picked up uniformly at random becomes an internal node and gives birth to $S + 1$ leaves of its same colour. Such a diagonal urn is called a Pólya-Eggenberger urn and has been long studied in the literature. We can for example cite this result by Athreya [Ath69] (for a complete proof, see [Ber06] or [CMP13]):

Theorem 10. *Let p and K be two positive integers and $(D_1(n), \dots, D_p(n))$ the composition vector at time n of an urn process of initial composition $\nu = {}^t(\nu_1, \dots, \nu_p)$ and with replacement matrix $(K+1)I_p$, then, asymptotically when n tends to infinity, almost surely,*

$$\frac{1}{nK} (D_1(n), \dots, D_p(n)) \rightarrow \mathbf{Z} = (Z_1, \dots, Z_p)$$

where \mathbf{Z} is a Dirichlet-distributed random vector of parameter $(\frac{\nu_1}{K}, \dots, \frac{\nu_p}{K})$.

Thus, projecting Equation (13) onto E via π_E , renormalising it by $n^{\lambda/S} \ln^\nu n$ and taking the limit when $n \rightarrow +\infty$ gives Theorem 9.

□

6 Densities

Via an analysis of Fourier transforms, we prove in this section that for all large eigenvalue λ , W^{CT} and W^{DT} both admit a density on \mathbb{C} . We generalize the method developed by Liu for smoothing equations with positive solutions and refer to [CLP13] for a similar proof in the case of m -ary trees, where a complex random variable happens to be solution of a fixed point equation. A similar result is also proved in [CPS11] or in [CMP13] for two colors urns, but in this particular case, the random variables W^{CT} and W^{DT} are real, and the proofs are thus different.

Theorem 11. *Under assumptions (B), (T) and (I):*

- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then the random variables W^{CT} and W^{DT} both admit a density on \mathbb{C} , and their support is the whole complex plane.
- If $\lambda \in \mathbb{R}$, then, the random variables W^{CT} and W^{DT} both admit a density on \mathbb{R} , and their support is the whole real line.

The strategy of the proof is the following. First focus on the discrete time, i.e. on W^{DT} :

- We prove that the support of $W_{\mathbf{e}_i}^{DT}$ contains some non-lattice set of points.
- It implies that the Fourier transform of W^{DT} is integrable and thus invertible (see Lemmas 5, 6 and 7). the variable W^{DT} has a density on \mathbb{C} .
- We finally deduce from the existence of a density and from Lemma 3 that the support of W^{DT} is the whole complex plane.

Via the martingale connection (3), it infers that, for all $c \in \{1, \dots, d\}$ $W_{\mathbf{e}_c}^{CT}$ has a density on \mathbb{C} . Theorems 9 and 5 permit to generalize to any initial composition.

Lemma 2. *There exists $i \in \{1, \dots, d\}$ such that $\mathbb{E}W_{\mathbf{e}_i}^{DT} \neq 0$.*

Proof. Recall that, for all $c \in \{1, \dots, d\}$,

$$W_{\mathbf{e}_c}^{DT} = \lim_{n \rightarrow +\infty} (1 + {}^tR)^{-1} \left(1 + \frac{{}^tR}{1 + S}\right)^{-1} \cdots \left(1 + \frac{{}^tR}{1 + (n-1)S}\right)^{-1} \pi_E(U_{\mathbf{e}_c}(n)),$$

as a martingale almost sure limit. Thus, $\mathbb{E}W_{\mathbf{e}_c}^{DT} = \pi_E(\mathbf{e}_c)$. Let us assume that $\mathbb{E}W_{\mathbf{e}_c}^{DT} = \pi_E(\mathbf{e}_c) = 0$, for all $c \in \{1, \dots, d\}$ it would imply that $\mathbf{e}_c \in \bigoplus_{F \neq E} F$ where F takes its values in the set of Jordan blocks of R , for all $c \in \{1, \dots, d\}$, which is impossible since $\mathbf{e}_1, \dots, \mathbf{e}_d$ is a basis of \mathbb{R}^d while $\bigoplus_{F \neq E} F$ has dimension at most $d-1$. □

Lemma 3. *For all $\mathbf{z} \in \times_{i=1}^d \text{Supp}(W_{\mathbf{e}_i}^{DT})$, for all $c \in \{1, \dots, d\}$, for all $(v_1, \dots, v_{\gamma_d^{(c)}})$ in the support of a Dirichlet- distributed random vector of parameter $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{\gamma_d^{(c)}})$, where $\omega_i = \theta_i/S$ for all $i \in \{1, \dots, c\}$, we have*

$$\sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} v_p^{\lambda/S} z_i \in \text{Supp}(W_{\mathbf{e}_c}^{DT}).$$

Proof. Remind that for a given complex random variable Z , for all $z \in \mathbb{C}$,

$$z \in \text{Supp}(Z) \Leftrightarrow \forall \varepsilon > 0, \mathbb{P}(|Z - z| < \varepsilon) > 0.$$

Let \mathbf{z} , let $(v_1, \dots, v_{\gamma_d^{(c)}})$ verifying the hypothesis of the lemma and let $\eta > 0$:

$$\begin{aligned} \left| W_{\mathbf{e}_c}^{DT} - \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} v_p^{\lambda/S} z_i \right| &\stackrel{(law)}{=} \left| \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)} \right)^{\lambda/S} W_{\mathbf{e}_i}^{(p)} - \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} v_p^{\lambda/S} z_i \right| \\ &\stackrel{(law)}{=} \left| \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)} \right)^{\lambda/S} (W_{\mathbf{e}_i}^{(p)} - z_i) - \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(v_p^{\lambda/S} - \left(V_p^{(c)} \right)^{\lambda/S} \right) z_i \right|. \end{aligned}$$

Thus, since $(z \mapsto z^{\lambda/S})$ is continuous on \mathbb{C} , we have, with positive probability,

$$\begin{aligned} \left| W_{\mathbf{e}_c}^{DT} - \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} v_p^{\lambda/S} z_i \right| &\leq \eta \sum_{p=1}^{\gamma_d^{(c)}} \left| \left(V_p^{(c)} \right)^{\lambda/S} \right| + \eta \sum_{i=1}^d z_i \\ &\leq (\gamma_d^{(c)} + \|z\|_1) \eta \leq (S + 1 + \|z\|_1) \eta, \end{aligned}$$

where $\|z\|_1 = \sum_{i=1}^d z_i$. For all $\varepsilon > 0$, we fix $\eta = \frac{\varepsilon}{S+1+\|z\|_1}$ to conclude the proof. \square

Lemma 4. *There exists a non zero z_0 such that, for all $t \in (0, 1)$,*

$$(t^{\lambda/S} + (1-t)^{\lambda/S}) z_0 \in \bigcap_{i=1}^d \text{Supp}(W_{\mathbf{e}_i}^{DT}).$$

Proof. Let us fix $c \in \{1, \dots, d\}$. Thanks to Lemma 2, there exists $z_0 \neq 0 \in \text{Supp}(W_{\mathbf{e}_c}^{DT})$. Applying Lemma 3, for all $i \in \{1, \dots, d\}$ such that $a_{c,i} \neq 0$, $z_0 \in \text{Supp}(W_{\mathbf{e}_i}^{DT})$. Since the urn is irreducible (Assumption (I)), we can thus conclude that, for all $i \in \{1, \dots, d\}$, $z_0 \in \text{Supp}(W_{\mathbf{e}_i}^{DT})$. Therefore, still by applying Lemma 3, for all $t \in [0, 1]$,

$$(t^{\lambda/S} + (1-t)^{\lambda/S}) z_0 \in \bigcap_{i=1}^d \text{Supp}(W_{\mathbf{e}_i}^{DT}).$$

\square

Corollary 1. *If $\lambda \in \mathbb{R}$, $\text{Supp}(W_{\mathbf{e}_c}^{DT}) = \mathbb{R}$ for all $c \in \{1, \dots, d\}$*

Proof. One just have to note that, in view of Lemma 4, there exists $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \subseteq \bigcap_{i=1}^d \text{Supp}(W_{\mathbf{e}_c}^{DT})$. Applying then Lemma 3 to $(v_1, \dots, v_{\gamma_d^{(c)}}) = \left(\frac{1}{\gamma_d^{(c)}}, \dots, \frac{1}{\gamma_d^{(c)}} \right)$ and $z_1 = \dots = z_d = z \in [-\varepsilon, \varepsilon]$ gives that, for all $z \in \bigcap_{i=1}^d \text{Supp}(W_{\mathbf{e}_c}^{DT})$, $(\gamma_d^{(c)})^{1-\lambda/S} \in \bigcap_{i=1}^d \text{Supp}(W_{\mathbf{e}_c}^{DT})$. Remark that $\gamma_d^{(c)} = \sum_{i=1}^d \tilde{a}_{c,i} + \delta_{c,i} > 1$, because we know that $\sum_{i=1}^d a_{c,i} = S > 1$, which ensures that at least one $a_{c,i}$ and thus at least one $\tilde{a}_{c,i}$ is non zero. Thus, the iteration of the mapping $z \mapsto (\gamma_d^{(c)})^{\lambda/S}$ maps $[-\varepsilon, \varepsilon]$ into \mathbb{R} , which implies the result. \square

The three following lemmas are proven via very similar arguments that the one developed in [CMP13]. There is no additional idea to the proof here, except being careful to the slight changes induced by the higher $d \geq 3$ and by the weaker assumption (T) instead of T_{-1} . For all $c \in \{1, \dots, d\}$, let $\phi_c(t) = \mathbb{E} e^{it W_{\mathbf{e}_c}^{DT}}$ for all $t \in \mathbb{C}$ and $\psi_c(r) = \sup_{|t|=r} |\phi_c(t)|$.

Lemma 5. *For all $c \in \{1, \dots, d\}$, for all $r > 0$, $\psi_c(r) < 1$.*

Proof. We know that $\psi_c(0) = 1$ and that $\psi_c(r) \geq 1$ for all $r \geq 0$. Let us assume that there exists $r_c > 0$ such that $\psi_c(r_c) = 1$. Then, there exists $z_c \in \mathbb{C}$ and $\theta_c \in \mathbb{R}$ such that $|z_c| = r_c$ and

$$\mathbb{E} e^{i \langle z_c, W_{\mathbf{e}_c}^{DT} \rangle} = e^{i \theta_c}.$$

Thus, the complex random variable $e^{i\langle z_c, W_{e_c}^{DT} \rangle - i\theta_c}$ has mean 1 and takes its values in the unit disc. It is therefore almost surely equal to 1, implying that almost surely, there exists a constant $k \in \mathbb{Z}$ such that $\langle z_c, W_{e_c}^{DT} \rangle = \theta_c + 2k\pi$. Therefore, for all $t \in [0, 1]$,

$$\langle (t^{\lambda/S} + (1-t)^{\lambda/S})z_0, z_c \rangle \in \theta_c + 2\pi\mathbb{Z}.$$

But, we have the following asymptotic behaviour when t tends to 0,

$$\langle (t^{\lambda/S} + (1-t)^{\lambda/S})z_0, z_c \rangle = \langle t^{\lambda/S} + (1-t)^{\lambda/S}, \bar{z}_0 z_c \rangle = \operatorname{Re}(\bar{z}_0 z_c) + \langle t^{\lambda/S}, \bar{z}_0 z_c \rangle + \mathcal{O}(t).$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\langle (t^{\lambda/S} + (1-t)^{\lambda/S})z_0, z_c \rangle$ varies continuously around $\operatorname{Re}(\bar{z}_0 z_c)$, which contradicts $\langle (t^{\lambda/S} + (1-t)^{\lambda/S})z_0, z_c \rangle \in \theta_c + 2\pi\mathbb{Z}$ for all $t \in [0, 1]$ and thus concludes the proof.

If $\lambda \in \mathbb{R}$, $\langle z_c, W_{e_c}^{DT} \rangle \in \theta_c + 2\pi\mathbb{Z}$ is directly impossible since $\operatorname{Supp}(W_{e_c}^{DT}) = \mathbb{R}$ (cf. Corollary 1). \square

Lemma 6. For all $c \in \{1, \dots, d\}$, $\lim_{r \rightarrow \infty} \psi_c(r) = 0$.

Proof. In view of Equation (11), for all $c \in \{1, \dots, d\}$,

$$\begin{aligned} \phi_c(t) &= \mathbb{E} \exp \left(it \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)} \right)^{\lambda/S} W_{e_i}^{(p)} \right) \\ &= \mathbb{E} \left[\mathbb{E} \left[\exp \left(it \sum_{i=1}^d \sum_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \left(V_p^{(c)} \right)^{\lambda/S} W_{e_i}^{(p)} \right) \middle| V_1, \dots, V_{S+1} \right] \right] \\ &= \mathbb{E} \prod_{i=1}^d \prod_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \mathbb{E} \left(e^{it(V_p^{(c)})^{\lambda/S} W_{e_i}^{(p)}} \middle| V_1, \dots, V_{S+1} \right) \\ &= \mathbb{E} \prod_{i=1}^d \prod_{p=\gamma_{i-1}^{(c)}+1}^{\gamma_i^{(c)}} \phi_i \left(\left(V_p^{(c)} \right)^{\lambda/S} t \right). \end{aligned}$$

Thus,

$$\phi_c(t) = \mathbb{E} \prod_{i=1}^d \phi_i(V_i^{\lambda/S} t)^{\tilde{a}_{c,i} + \delta_{c,i}},$$

where V_i is a Beta- $(\frac{\theta_i}{S}, 1)$ distributed random variable. Via Fatou's Lemma, it implies

$$\limsup_{r \rightarrow +\infty} \psi_c(r) \leq \mathbb{E} \prod_{i=1}^d (\limsup_{r \rightarrow +\infty} \psi_i(V_i^{\lambda/S} r))^{\tilde{a}_{c,i} + \delta_{c,i}}.$$

If we denote by $\ell_c = \limsup_{r \rightarrow +\infty} \psi_c(r)$, then, for all $c \in \{1, \dots, d\}$,

$$\ell_c \leq \prod_{i=1}^d \ell_i^{\tilde{a}_{c,i} + \delta_{c,i}}, \quad (14)$$

and $\ell_c \geq 1$.

If there exists $i \in \{1, \dots, d\}$ such that $\ell_i < 1$, then for all $c \in \{1, \dots, d\} \setminus \{i\}$, in view of equation (14), we have $a_{c,i} = 0$ or $\ell_c = 0$. Since the urn is assumed irreducible (Assumption (I)), i is a dominating colour (df. Definition 2), which implies, for all $c \in \{1, \dots, d\}$, $\ell_c = 0$. Thus, either $\ell_c = 1$ for all $c \in \{1, \dots, d\}$, either $\ell_c = 1$ for all $c \in \{1, \dots, d\}$. Let us assume that $\ell_c = 1$ for all $c \in \{1, \dots, d\}$.

Let us define $\psi(r) = \max_{c=1, \dots, d} \psi_c(r)$. According to Lemma 5, for all $c \in \{1, \dots, d\}$, $\psi_c(1) < 1$, and thus, $\psi(1) < 1$. Let $\varepsilon \in]0, 1 - \psi(1)[$, and define:

$$\begin{aligned} r_1(\varepsilon) &= \max\{r \in]0, 1[, \psi(r) = 1 - \varepsilon\} \\ r_2(\varepsilon) &= \min\{r > 1, \psi(r) = 1 - \varepsilon\}. \end{aligned}$$

These definitions are legal since ψ is continuous, $\psi(0) = 1$ and $\limsup_{r \rightarrow \infty} \psi(r) = 1$. Moreover, we have $\psi(r_1(\varepsilon)) = \psi(r_2(\varepsilon)) = 1 - \varepsilon$, and for all $r \in [r_1(\varepsilon), r_2(\varepsilon)]$, $\psi(r) \leq 1 - \varepsilon$. Let ρ be an adherent point of $(r_1(\varepsilon))_{0 < \varepsilon < 1}$. Then, $\psi(\rho) = 1$, which implies $\rho = 0$: if $r_1(\varepsilon)$ converges when ε tends to 0, then the limit has to be 0. We know otherwise that for all $c \in \{1, \dots, d\}$

$$\psi_c(r) \leq \mathbb{E} \prod_{i=1}^d \psi_i(|V_i^{\lambda/S}|r)^{\tilde{a}_{c,i} + \delta_{c,i}},$$

where V_i is a Beta-distributed random variable of parameter $(\frac{\theta_i}{S}, 1)$. In particular, for all $r \geq 0$,

$$\psi_c(r) \leq \psi_c(|V_c^{\lambda/S}|r).$$

Iterating this identity, we get

$$\psi(r) \leq \mathbb{E} \psi(r A_1^{(c)} \dots A_n^{(c)})$$

where $(A_i^{(c)})_{i \geq 1}$ is a sequence of i.i.d. random variables having the same law as $|V_c^{\lambda/S}|$. Let us define, for all $c \in \{1, \dots, d\}$,

$$\lambda_n^{(c)}(r, \varepsilon) = \mathbb{P}(r_1(\varepsilon) \leq r A_1^{(c)} \dots A_n^{(c)} \leq r_2(\varepsilon)).$$

For all $r \geq 0$, for all $c \in \{1, \dots, d\}$,

$$\psi_c(r) \leq 1 - \varepsilon \lambda_n^{(c)}(r, \varepsilon),$$

which implies,

$$\psi(r) \leq 1 - \varepsilon \lambda_n(r, \varepsilon),$$

where $\lambda_n(r, \varepsilon) = \min_{i \in \{1, \dots, d\}} \lambda_n^{(i)}(r, \varepsilon)$. For all $c \in \{1, \dots, d\}$, we have,

$$\begin{aligned} \psi_c(r_2(\varepsilon)) &\leq \mathbb{E} \prod_{i=1}^d \psi_i(|V_i^{\lambda/S}|r_2(\varepsilon))^{\tilde{a}_{c,i} + \delta_{c,i}} \\ &\leq \mathbb{E} \prod_{i=1}^d (1 - \varepsilon \lambda_n^{(i)}(A^{(i)}r_2(\varepsilon), \varepsilon))^{\tilde{a}_{c,i} + \delta_{c,i}} \end{aligned}$$

where $A^{(i)} \stackrel{(law)}{=} |V_i^{\lambda/S}|$. Remark that, for all $c \in \{1, \dots, d\}$,

$$\lambda_n^{(c)}(r_2(\varepsilon)A^{(c)}, \varepsilon) \rightarrow \mu_n(A^{(c)}) := \mathbb{P}(0 \leq A^{(i)}A_1^{(i)} \dots A_n^{(i)} \leq 1).$$

Thus, for all $c \in \{1, \dots, d\}$,

$$\frac{1 - \mathbb{E} \prod_{i=1}^d (1 - \varepsilon \lambda_n^{(i)}(A^{(i)}r_2(\varepsilon), \varepsilon))^{\tilde{a}_{c,i} + \delta_{c,i}}}{\varepsilon} \rightarrow \sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \mu_n(A^{(i)}).$$

Now remark that

$$1 - \varepsilon = \psi(r_2(\varepsilon)) = \max_{c \in \{1, \dots, d\}} \psi_c(r_2(\varepsilon)) \leq \max_{c \in \{1, \dots, d\}} \mathbb{E} \prod_{i=1}^d (1 - \varepsilon \lambda_n^{(i)}(A^{(i)}r_2(\varepsilon), \varepsilon))^{\tilde{a}_{c,i} + \delta_{c,i}},$$

which implies that

$$\min_{c \in \{1, \dots, d\}} \sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \mu_n(A^{(i)}) \leq 1.$$

Let us now choose $m \in \{1, \dots, d\}$ such that $\theta_m = \max_{i \in \{1, \dots, d\}} \theta_i$. Thus, for all $i \in \{1, \dots, d\}$, V_i is stochastically less than V_m and thus, $A^{(i)}$ is stochastically less than $A^{(m)}$, which implies that, for all $c \in \{1, \dots, d\}$,

$$2\mu_n(A^{(m)}) \leq \min_{c \in \{1, \dots, d\}} \gamma_d^{(c)} \mu_n(A^{(m)}) \leq 1. \quad (15)$$

Via Markov inequality, we get: $1 - \mu_n(x) \leq x \mathbb{E}(A_1^{(m)} \dots A_n^{(m)}) = x(\mathbb{E}A^{(m)})^n$. Remark also that

$$\mathbb{E}|V_m^{\lambda/S}| = \mathbb{E} \left| V_m^{\frac{\text{Re}\lambda}{S} + i \frac{\text{Im}\lambda}{S}} \right| = \mathbb{E} V_m^{\text{Re}\lambda/S} = \frac{1}{1 + \frac{\text{Re}\lambda}{\theta_m S}} < 1$$

since $\frac{\operatorname{Re}\lambda}{\theta_m S} > 0$. Thus, we can conclude that $\lim_{n \rightarrow +\infty} \mu_n(x) = 1$, and via dominated convergence,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \mu_n \left(A^{(m)} \right) = 1. \quad (16)$$

Thus Equations (15) and (16) cannot both hold, which thus proves Lemma 6. \square

Lemma 7. *For all $c \in \{1, \dots, d\}$, for all $\rho \in (0, \theta_c / \operatorname{Re}\lambda)$, asymptotically when $|t|$ tends to $+\infty$, $\phi_c(t) = \mathcal{O}(t^{-\rho})$.*

Proof. Let $\varepsilon > 0$. In view of Lemma 6, there exists $T > 0$ such that, for all $|t| \geq T$, for all $i \in \{1, \dots, d\}$, $|\phi_i(t)| \leq \varepsilon$. We already have proved that

$$|\phi_c(t)| = \mathbb{E} \prod_{i=1}^d \prod_{p=\gamma_{i-1}^{(c)}}^{\gamma_i^{(c)}} \left| \phi_i \left(\left(V_p^{(c)} \right)^{\lambda/S} t \right) \right|.$$

Thus, for all $t \in \mathbb{R}$,

$$|\phi_c(t)| \leq \varepsilon^{\gamma_d^{(c)}-1} \mathbb{E} |\phi_c(V_{S+1}^{\lambda/S} t)| + \sum_{p=1}^{\gamma_d^{(c)}} \mathbb{P} \left(\left| \left(V_p^{(c)} \right)^{\frac{\lambda}{S}} t \right| < T \right),$$

where m is chosen such that $\theta_m = \min_{i \in \{1, \dots, d\}} \theta_i$. Recall that $V_m \stackrel{(law)}{=} U^{\lambda/\theta_m}$ where U is a uniform random variable on $(0, 1)$. Thus,

$$|\phi_c(t)| \leq \varepsilon^{\gamma_d^{(c)}-1} \mathbb{E} |\phi_c(U^{\lambda/\theta_c} t)| + \gamma_d^{(c)} \left(\frac{T}{|t|} \right)^{\theta_m / \operatorname{Re}\lambda},$$

for all $t \in \mathbb{R} \setminus \{0\}$. Said differently, there exists a positive constant C such that, for all non zero real t ,

$$|\phi_c(t)| \leq \varepsilon^{\gamma_d^{(c)}-1} \mathbb{E} |\phi_c(U^{\lambda/\theta_m} t)| + C \left(\frac{1}{|t|} \right)^\rho,$$

For all $\rho \in (0, \theta_c / \operatorname{Re}\lambda)$, $\mathbb{E} U^{-\frac{\rho \operatorname{Re}\lambda}{\theta_c}} < +\infty$, and we can thus apply a Gronwall-type lemma ([Liu99, Lemma 4.1]), which implies

$$|\phi_c(t)| \leq \frac{C |t|^{-\rho}}{1 - \varepsilon^{\gamma_d^{(c)}-1} \mathbb{E} U^{-\frac{\rho \operatorname{Re}\lambda}{\theta_c}}},$$

provided that ε verifies $1 - \varepsilon^{\gamma_d^{(c)}-1} \mathbb{E} U^{-\frac{\rho \operatorname{Re}\lambda}{\theta_c}} > 0$. \square

Proposition 2. • *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the distribution of $W_{e_c}^{DT}$ admits a density on \mathbb{C} and its support is \mathbb{C} , for all $c \in \{1, \dots, d\}$, and their support in the whole complex plane.*

- *If $\lambda \in \mathbb{R}$, the distribution of W^{DT} admits a density on \mathbb{R} and its support is \mathbb{R} , for all $c \in \{1, \dots, d\}$, and their support is the whole real line.*

Proof. First assume that $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let us apply arguments already used in [CMP13, page 22] to prove that ϕ_c is integrable for all $c \in \{1, \dots, d\}$. Recall that, for all $t \in \mathbb{C}$,

$$|\phi_c(t)| \leq \mathbb{E} \prod_{i=1}^d \left| \phi_i \left(V_i^{\lambda/S} t \right) \right|^{\bar{a}_{c,i} + \delta_{c,i}}.$$

In view of Lemma 7, there exists a constant $\kappa > 0$ such that, for all $i \in \{1, \dots, d\}$, for all $\rho_i \in [0, \theta_i / \operatorname{Re}\lambda)$, for all $|t|$ large enough, $|\phi_i(t)| \leq \kappa |t|^{-\rho_i}$. Thus, for all $|t|$ large enough,

$$|\phi_c(t)| \leq \frac{\kappa^{\gamma_d^{(c)}}}{|t|^{\sum_{i=1}^d (\bar{a}_{c,i} + \delta_{c,i}) \rho_i}} \mathbb{E} \prod_{i=1}^d \left| V_i^{-\frac{\lambda \rho_i}{S}} \right|.$$

Let $\eta_c = \sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \rho_i$, we have, by using a formula for joint moments of Dirichlet random vectors (see for example [CMP13, page 34]), we have

$$|\phi_c(t)| \leq \frac{\kappa \gamma_d^{(c)}}{|t|^{\eta_c}} \frac{\Gamma(1 + \theta_c/S)}{\Gamma(1 + \theta_c/S - \frac{\operatorname{Re} \lambda \eta_c}{S})} \prod_{i=1}^d \frac{\Gamma((\tilde{a}_{c,i} + \delta_{c,i}) \frac{\theta_i - \operatorname{Re} \lambda \rho_i}{S})}{\Gamma((\tilde{a}_{c,i} + \delta_{c,i}) \frac{\theta_i}{S})}.$$

Remark that since $\rho < \theta_m/S \leq \theta_c/S$, we have that

$$1 + \theta_c/S - \frac{\operatorname{Re} \lambda \eta_c}{S} \geq 1 + \theta_c/S \left(1 - \frac{\operatorname{Re} \lambda \gamma_d^{(c)}}{S}\right) > 1 + \theta_c/S (1 - \gamma_d^{(c)}) \geq 0,$$

by assumptions (B) and because the eigenvalue S is a simple eigenvalue of R (in particular, $\lambda \neq S$). Thus,

$$|\phi_c(t)| = \mathcal{O}(|t|^{-\eta_c}),$$

for all $\eta_c = \sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \rho_i$ with $\rho_i \in [0, \theta_i/S]$. Since

$$\sum_{i=1}^d (\tilde{a}_{c,i} + \delta_{c,i}) \theta_i/S = 1 + \frac{\theta_c}{S} > 1,$$

it implies that ϕ_c is integrable and thus that $W_{e_c}^{DT}$ admits a bounded and continuous density on \mathbb{C} , for all $c \in \{1, \dots, d\}$.

This result implies that, there exists some $\varepsilon > 0$ such that, for all $c \in \{1, \dots, d\}$, there exists $z_c \in \mathbb{C}$ such that the opened ball centred in z_c of radius ε is contained into $\operatorname{Supp}(W_{e_c}^{DT})$. In view of Lemma 3, for all $c, i \in \{1, \dots, d\}$ such that $\tilde{a}_{c,i} + \delta_{c,i} \geq 1$,

$$\operatorname{Supp}(W_{e_i}^{DT}) \subseteq \operatorname{Supp}(W_{e_c}^{DT}).$$

The fact that the considered urn is irreducible tells us that thus, there exists $z_0 \in \mathbb{C}$ such that the ball centered in z_0 of radius ε is included into $\bigcap_{i \in \{1, \dots, d\}} \operatorname{Supp}(W_{e_i}^{DT})$. By applying again Lemma 3 to the vector $(\frac{1}{\gamma_d^{(c)}}, \dots, \frac{1}{\gamma_d^{(c)}})$, we thus get that, for all $c \in \{1, \dots, d\}$, for all $z \in \bigcap_{i \in \{1, \dots, d\}} \operatorname{Supp}(W_{e_i}^{DT})$,

$$(\gamma_d^{(c)})^{1-\lambda/S} z \in \operatorname{Supp}(W_{e_c}^{DT}).$$

Since the iterated mapping $(z \mapsto (\gamma_d^{(c)})^{1-\lambda/S} z)$ maps $\mathcal{B}(z_0, \varepsilon)$ into \mathbb{C} (since $S > 0$ and $\operatorname{Re} \lambda < S$), then the support of $W_{e_c}^{DT}$ is the whole complex plane.

In the case $\lambda \in \mathbb{R}$, the proof of the existence of a density is exactly the one developed in [CMP13]: ϕ_c is integrable, which implies that $W_{e_c}^{DT}$ admits a bounded and continuous density on \mathbb{R} , for all $c \in \{1, \dots, d\}$. \square

Proof of Theorem 11. Connexion (3) permits to transfer results stated in Proposition 2 to $W_{e_c}^{CT}$ for all $c \in \{1, \dots, d\}$, since ξ and $W_{e_c}^{DT}$ both admit a density and are independent. Finally, Theorems 9 and 5 allow to generalize to any initial composition α . \square

Remark: For two-colour Pólya urns, it is proven under (B), (I) and (T_{-1}) [CPS11] that the density of W_{α}^{CT} explodes in zero for all initial composition α . Is it possible to prove the same result for d -colour urns? We believe that the proof is similar to the one developed in [CPS11] if λ is real, and under (T) instead of (T_{-1}) , but what can be said about W^{CT} in zero when λ is complex?

7 Conclusion and perspectives

This paper contains the first results concerning the study of the W random variables exhibited by the asymptotic behaviour of a multi-colour Pólya urn process projected along large Jordan blocks.

It is interesting to see how the methods used in [CMP13], especially the forest representation, can be adapted quite straightforwardly to multi-colour urn processes. The main difficulty is that the

W are complex random variables in this multi-colour case, but the moments and Fourier transform analysis stay overall identical the two-colour case. A major difference with the two-colour case is that we have to consider possibly negative diagonal coefficients in the replacement matrix: we show how to handle such diagonal coefficients (as soon as a tenability hypothesis is made), and show how the forest representation can be adapted to such Pólya urns.

However, many questions still remain about the random variables W : some of them are solved in the two-colour case (by methods that do not seem easy to generalize to multi-colour urns) but other are not proved even in the two-colour case. Can we find a lower bound for the moments of the W ([CPS11] for two-colour urns)? What is the exact order of their moments? Is the Laplace transform of W^{CT} convergent (this is false in the two-colour case [CPS11])? Is the Fourier transform of W^{CT} integrable (also false in the two-colour case [CPS11])? Is the density of W^{CT} continuous (also false in two-colour [CPS11])? Do the W have heavy tails?

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